Avalanche Photodiode Statistics in Triggered-Avalanche Detection Mode

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The output of a triggered-avalanche mode avalanche photodiode is modeled as Poisson-distributed primary avalanche events plus conditionally Poisson-distributed trapped-carrier-induced secondary events. The moment generating function as well as the mean and variance of the diode output statistics are derived. The dispersion of the output statistics is shown to always exceed that of the Poisson distribution. Several examples are considered in detail.

I. Introduction

When a sufficiently large enough voltage is applied across an avalanche photodiode (APD), free carriers accelerated by the electric field attain enough energy to generate secondary carrier pairs by impact ionization. This leads to carrier avalanche and at a high enough voltage to avalanche breakdown. The normal operating mode of APDs biased below the avalanche breakdown voltage does not, however, have sufficiently favorable noise characteristics for photon counting applications. Recent work (Refs. 1 through 3) has indicated that the APD operated in a triggered-avalanche detection (TAD) mode might be effective for photon-counting applications.

Suppose initially an APD is biased below avalanche breakdown and is sufficiently cooled and shielded to eliminate all free carriers in the APD junction region. The bias voltage is then increased above the avalanche breakdown voltage threshold, but below the zener breakdown voltage. In the absence of free carriers in the junction, no avalanche breakdown can occur. The presence of a single free carrier, for example as a result of photon absorption, can initiate a self-sustaining avalanche breakdown in which the current flow in the diode grows exponentially until the bias voltage is reduced for a sufficient period of time to quench the avalanche discharge and sweep all the free carriers from the diode junction. The large bias can then be reapplied so that the diode can again detect photons. This is called a triggered-avalanche detection (TAD) mode (Refs. 1 and 2) or a Geiger-counter mode (Ref. 3) of operation.

The TAD-mode output pulses are sufficiently large so that no additional amplification is necessary to count them. Hence, single-photon detection is feasible. Moreover, TAD-mode APDs can have significant quantum efficiency improvements over the photomultiplier tubes commonly used for photon counting. TAD-mode APDs, however, have a problem with carrier trapping. During each avalanche, carriers can become trapped at dislocations and impurities within the diode. Those trapped carriers not swept out during the quenching period can subsequently initiate avalanches when they are released

from the traps. Experimental results give evidence of trap lifetimes of up to minutes in duration (Refs. 1 to 3) over which carriers are released. It appears that an evaluation of the photon counting capability of the TAD-mode APD has to include the effect of carrier-induced avalanches.

This article considers a simple and most possibly naive statistical model to account for carrier trapping in a TAD-mode APD. This model is then analyzed to obtain the statistics of the diode output. The moment generating function of the diode output statistics is derived in Section II along with the mean and variance. Specific examples are considered in Section III and a discussion of the results is given in Section IV.

II. Moment Generating Function of Count Statistics

Avalanche discharges in a TAD-mode APD are initiated either by thermally or photon-absorption-induced free carriers or by trap-released free carriers. The first two types of free-carrier-induced discharges can be regarded as primary discharge events. Since trapped carriers exist as a result of previous avalanche discharges, the discharges generated by these carriers can be regarded as secondary events. We shall make the assumption here that trapped-carrier-induced secondary avalanche discharges do not produce more trapped carriers. That is, all the trapped carriers in the APD were generated by previous primary avalanche discharge events. This simplifying assumption is made to make the subsequent analysis tractable.

To describe the diode output statistics, let us define for each $t > 0$, the counting processes:

$N_1(t)$ = number of primary avalanche events due to either thermally or photon-induced free carriers occurring in a time interval $[0, t)$.

$N_2(t)$ = number of secondary trapped-carrier-induced avalanche events in a time interval $[0, t)$.

It can be assumed that $N_1(t)$ is a Poisson process with intensity rate $\lambda(t)$ equal to the average rate at which primary avalanche discharge events occur. This intensity rate $\lambda(t)$ typically includes a constant rate at which thermally generated carriers initiate avalanche discharges and a rate at which photoelectron-initiated avalanches occur on the average.

We assume next that each primary avalanche event occurring at time τ will generate secondary avalanche events in the time interval $[\tau, \infty)$ at an average rate $h(t - \tau)$. It is assumed that this rate function $h(t)$ of generating trapped-carrier avalanche events by a single primary event is fixed and deterministic. So, if $\{t_i\}$ are the occurrence times of the primary avalanche events, then conditioned on the counting process

$N_1(t)$, $N_2(t)$ is a conditional Poisson process with intensity rate $\lambda_2(t)$ given by

$$\lambda_2(t) = \sum_{i=1}^{N(t)} h(t - t_i), \qquad t \geqslant 0 \qquad (1)$$

Since $\lambda_2(t)$ is a random process, $N_2(t)$ is effectively a Poisson process with random intensity rate. Such processes are called doubly stochastic Poisson processes (Ref. 4). Moreover, $\lambda_2(t)$ is a filtered Poisson process or shot-noise process (Ref. 4) with filter impulse response $h(t)$.

The process $N_1(t) + N_2(t)$ then gives the total number of avalanche events in the TAD-mode APD output during the time period $[0, t)$. It is interesting to determine the statistics of the TAD-mode APD output in some time interval after the device has been operating for a long period of time. To determine such statistics, let $T > 0$ and $\Delta T > 0$ and define

ΔN_1 = $N_1(T + \Delta T) - N_1(T)$
 = number of primary avalanche events in the time interval $[T, T + \Delta T)$

ΔN_2 = $N_2(T + \Delta T) - N_2(T)$
 = number of secondary avalanche events in the time interval $[T, T + \Delta T)$

The random variables of interest are then

$\Delta N_1 + \Delta N_2$ = total number of avalanche events in the TAD-mode APD in the time interval $[T, T + \Delta T)$

Assume that the device started operation at $t = 0$. Then $\Delta N_1 + \Delta N_2$ represents the number of avalanche events in the TAD-mode APD output during a time interval of length ΔT after the device has already been operating for a time period of length T. The case of particular interest is then when T is significantly larger than ΔT.

Teich and his collaborators (Ref. 5) have studied doubly stochastic Poisson processes such as $N_2(t)$. They have determined the moment generating function and several of the moments of $\Delta N_1 + \Delta N_2$ only in the case when $T = 0$ and hence are not useful here. Our goal is to determine the moment generating function (mgf)

$$\phi(s) = E\left[e^{-s(\Delta N_1 + \Delta N_2)}\right] \qquad (2)$$

of $\Delta N_1 + \Delta N_2$ for general $\lambda(t)$ and $h(t)$. The moments of $\Delta N_1 + \Delta N_2$ can also be determined using this mgf. A detailed derivation given in Appendix A shows that

$$\phi(s) = \phi_1(s) \cdot \phi_2(s) \qquad (3)$$

where

$$\phi_1(s) = \exp\left\{\int_0^T \lambda(\tau)\right.$$

$$\left. \times \left[\exp\left\{(e^{-s}-1)\int_T^{T+\Delta T} h(t-\tau)\,dt\right\} - 1\right]d\tau\right\} \quad (4)$$

and

$$\phi_2(s) = \exp\left\{\int_T^{T+\Delta T} \lambda(\tau)\right.$$

$$\left. \times \left[e^{-s}\exp\left\{(e^{-s}-1)\int_T^{T+\Delta T} h(t-\tau)\,dt\right\} - 1\right]d\tau\right\} \quad (5)$$

To understand the significance of $\phi_1(s)$ and $\phi_2(s)$, consider the following special cases. First, suppose that $\lambda(t) = 0$ for $0 \leqslant t \leqslant T$. Then $\phi(s) = \phi_2(s)$. In this case there are no primary events in the time interval $[0, T)$. Hence, we can conclude that $\phi_2(s)$ is the mgf of the sum of the primary events in the time interval $[T, T + \Delta T)$ plus the secondary events in this time period caused by these primaries. Next, consider when $\lambda(t) = 0$ for $T \leqslant t \leqslant T + \Delta T$. Then $\phi(s) = \phi_1(s)$. In this case, there are no primary events in the time interval $[T, T + \Delta T)$. Hence $\phi_1(s)$ is the mgf of the secondary events in the time interval $[T, T + \Delta T)$ caused by the primary events in the interval $[0, T)$. Since $\phi_1(s)$ depends only on $\lambda(t)$ for $0 \leqslant t \leqslant T$ and $\phi_2(s)$ depends only on $\lambda(t)$ for $T \leqslant t \leqslant T + \Delta T$, these two conclusions hold in general. Also, note the statistical independence of these two count sums, which is a result of the independent increments property of $N_1(t)$.

Finally, derivatives of $\phi(s)$ at $s = 0$ can be determined to obtain the moments of $\Delta N_1 + \Delta N_2$. It can be shown that

$$-\phi_1'(0) = \int_0^T \lambda(\tau)\left[\int_T^{T+\Delta T} h(t-\tau)\,dt\right]d\tau \quad (6)$$

$$\phi_1''(0) = \left[\phi_1'(0)\right]^2 + \int_0^T \lambda(\tau)$$

$$\times \left[\int_T^{T+\Delta T} h(-\tau)\,dt + \left(\int_T^{T+\Delta T} h(t-\tau)\,dt\right)^2\right]d\tau \quad (7)$$

$$-\phi_2'(0) = \left[\int_T^{T+\Delta T} \lambda(\tau)\int_T^{T+\Delta T} h(t-\tau)\,dt\right]d\tau$$

$$+ \int_T^{T+\Delta T} \lambda(\tau)\,d\tau \quad (8)$$

$$-\phi_2''(0) = \left[\phi_2'(0)\right]^2 + \int_T^{T+\Delta T} \lambda(\tau)$$

$$\times \left[\int_T^{T+\Delta T} h(t-\tau)\,dt + \left(\int_T^{T+\Delta T} h(t-\tau)\,dt\right)^2\right]d\tau$$

$$+ 2\int_T^{T+\Delta T} \lambda(\tau)\left[\int_T^{T+\Delta T} h(t-\tau)\,dt\right]d\tau$$

$$+ \int_T^{T+\Delta T} \lambda(\tau)\,d\tau \quad (9)$$

Hence, it follows from Eqs. (3), (6), and (8) that the mean of $\Delta N_1 + \Delta N_2$ is given by

$$E[\Delta N_1 + \Delta N_2] = \int_T^{T+\Delta T} \lambda(\tau)\left[\int_T^{T+\Delta T} h(t-\tau)\,dt\right]d\tau$$

$$+ \int_T^{T+\Delta T} \lambda(\tau)\,d\tau \quad (10)$$

Since

$$\int_T^{T+\Delta T} \lambda(\tau)\,d\tau$$

is the average number of primary events in the time interval $[T, T + \Delta T)$, it follows that the first term in Eq. (10) is the average number of secondary events in that interval. Moreover, Eq. (6) gives the average number of secondaries in $[T, T + \Delta T)$ that are a result of primaries in $[0, T)$.

The variance of $\Delta N_1 + \Delta N_2$ is obtained from Eqs. (3), (7), and (9) and is given by

$$\text{var}\,(\Delta N_1 + \Delta N_2) = \int_0^{T+\Delta T} \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt \right.$$

$$\left. + \left(\int_T^{T+\Delta T} h(t-\tau)\,dt \right)^2 \right] d\tau$$

$$+ 2 \int_T^{T+\Delta T} \lambda(\tau)$$

$$\times \left[\int_T^{T+\Delta T} h(t-\tau)\,dt \right] d\tau$$

$$+ \int_T^{T+\Delta T} \lambda(\tau)\,d\tau \qquad (11)$$

Note that

$$\int_T^{T+\Delta T} \lambda(\tau)\,d\tau$$

is the variance of the primary count in $[T,\ T+T)$. Note from Eq. (7) that the variance of the secondary count in $[T,\ T+T)$ due to primaries in $[0,\ T)$ is given by

$$\int_0^T \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt + \left(\int_T^{T+\Delta T} h(t-\tau)\,dt \right)^2 \right] d\tau \qquad (12)$$

Hence, it follows that

$$\int_T^{T+\Delta T} \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt + \left(\int_T^{T+\Delta T} h(t-\tau)\,dt \right)^2 \right] d\tau \qquad (13)$$

must be the variance of the secondary count in $[T,\ T+\Delta T)$ due to primaries in that same interval. So the first term in Eq. (11) represents the variance of the total secondary count in $[T,\ T+\Delta T)$. Moreover, an examination of Eqs. (9) and (13) yields the obvious conclusion that the second term in Eq. (11) is the covariance between the primary and secondary counts in $[T,\ T+\Delta T)$.

Another interesting statistic for counting distributions is the dispersion or variance-to-mean ratio of the distribution. The dispersion of a Poisson counting distribution is equal to one. It is then of some interest to compare dispersions with that of the Poisson distribution. Here

$$\frac{\text{var}\,(\Delta N_1 + \Delta N_2)}{E[\Delta N_1 + \Delta N_2]} = 1 + \frac{\int_0^{T+\Delta T} \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt \right]^2 d\tau + 2 \int_T^{T+\Delta T} \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt \right] dt}{\int_0^{T+\Delta T} \lambda(\tau) \left[\int_T^{T+\Delta T} h(t-\tau)\,dt \right] d\tau + \int_T^{T+\Delta T} \lambda(\tau)\,d\tau} \qquad (14)$$

Hence, the quotient term in Eq. (14) gives the excess dispersion over the dispersion of the Poisson distribution. This implies that the distribution of $\Delta N_1 + \Delta N_2$ has a wider spread about its mean than the Poisson distribution.

III. Examples

Example 1: Constant Light Source and Constant Rate of Trapped Carrier Induced Events

Consider when

$$\lambda(t) = \bar{\lambda} = \text{constant}$$

and

$$h(t) = \begin{cases} A, & 0 \leq t \leq T_h \\ 0, & \text{otherwise} \end{cases}$$

where

$$\Delta T \leq T_h \leq T \ (\text{see Fig. 1}) \qquad (15)$$

An average constant rate $\bar{\lambda}$ of primary avalanche events per unit time is generated here as a result of either photoelectrons or thermally induced free carriers in the diode. Moreover, it

is assumed that each primary avalanche event will, on the average, generate trapped induced secondary events at a constant rate A per unit time over a time period of length T_h. We assume that $T \geqslant T_h$ so that the diode has reached a steady-state condition. We also assume for convenience that the avalanche event counting interval $\Delta T \leqslant T_h$. Experimental work (Refs. 1 and 3) has reported T_h ranging from a few milliseconds to minutes. Since $N_2(t)$ is a doubly stochastic Poisson process, AT_h represents the average number of secondary avalanche events generated per primary event. There is experimental work (Ref. 3) that indicates that $AT_h = 0.1$ may be achieved with commercial APDs.

For this example, it is easy to show that

$$
\int_T^{T+\Delta T} h(t-\tau)\,dt =
\begin{cases}
0, & 0 \leqslant \tau \leqslant T - T_h \\[1ex]
A(\tau - T + T_h), & T - T_h \leqslant \tau \leqslant T - T_h + \Delta T \\[1ex]
A(\Delta T), & T - T_h + \Delta T \leqslant \tau \leqslant T \\[1ex]
A(T + \Delta T - \tau), & T \leqslant \tau \leqslant T + \Delta T
\end{cases}
\tag{16}
$$

Substitution of Eq. (16) into Eqs. (3) through (5) yields

$$
\phi(s) = \exp\left\{ \frac{\bar{\lambda}\,[\exp\{(e^{-s}-1)A(\Delta T)\}-1]}{A(e^{-s}-1)} - \bar{\lambda}(\Delta T)\right\}
$$

$$
\cdot \exp\{\bar{\lambda}(T_h - \Delta T)\,[\exp\{(e^{-s}-1)A(\Delta T)\}-1]\}
$$

$$
\cdot \exp\left\{ \frac{\bar{\lambda}\,e^{-s}\,[\exp\{(e^{-s}-1)A(\Delta T)\}-1]}{A(e^{-s}-1)} - \bar{\lambda}(\Delta T)\right\}
\tag{17}
$$

The product of the first two exponential factors in Eq. (17) is the mgf $\phi_1(s)$ and the last exponential factor is the mgf $\phi_2(s)$. The second exponential factor turns out to be the mgf of a two-parameter Neyman Type-A probability distribution (Ref. 6). Hence all three exponential factors in Eq. (17) are mgfs.

Substitution of Eq. (16) into Eqs. (10) and (11) yields

$$
E[\Delta N_1 + \Delta N_2] = \bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)
\tag{18}
$$

$$
\text{var}(\Delta N_1 + \Delta N_2) = \bar{\lambda}T_h(A\,\Delta T) + \bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)^2
$$
$$
+ \bar{\lambda}\Delta T(A\,\Delta T) - \frac{\bar{\lambda}\Delta T(A\,\Delta T)^3}{3}
\tag{19}
$$

The first term in Eq. (18) is the average number of primaries and the second term is the average number of secondaries. Note that the average number of secondaries is equal to (the average number of primaries over a time interval of length T_h) $\times$ (average number of secondaries per primary) $= (\bar{\lambda}T_h) \times (A\,\Delta T)$. This is about $(\bar{\lambda}\Delta T)\,(A\,\Delta T)$ less than one would expect at a first glance. The loss is due to edge effects as evidenced in Eq. (16). The dispersion is given by

$$
\frac{\text{var}(\Delta N_1 + \Delta N_2)}{E[\Delta N_1 + \Delta N_2]} =
$$

$$
1 + \frac{\bar{\lambda}T_h(A\,\Delta T)^2 + \bar{\lambda}\Delta T(A\,\Delta T) - \dfrac{(\bar{\lambda}\Delta T)(A\,\Delta T)^2}{3}}{\bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)}
\tag{20}
$$

The excess dispersion over the dispersion of the Poisson distribution is of the order $A\,\Delta T + O(\Delta T)^2$ as $\Delta T \to 0$. So, as $\Delta T \to 0$, the dispersion approaches that of the Poisson distribution. In fact, the distribution of $\Delta N_1 + \Delta N_2$ tends to a Poisson distribution as $\Delta T \to 0$. To show this, assume that ΔT is small so that

$$
\exp\{(e^{-s}-1)A(\Delta T)\} \approx
$$

$$
1 + (e^{-s}-1)A(\Delta T) + \frac{(e^{-s}-1)^2 A^2(\Delta T)^2}{2}
\tag{21}
$$

Substitution of the approximation (21) into Eq. (17) yields

$$
\phi(s) \approx \exp\left\{ \bar{\lambda}(e^{-s}-1)(\Delta T) + \frac{\bar{\lambda}e^{-s}(e^{-s}-1)A(\Delta T)^2}{2}\right\}
$$

$$
\cdot \exp\left\{ \frac{\bar{\lambda}(e^{-s}-1)A(\Delta T)^2}{2}\right\}
$$

$$
\cdot \exp\left\{ \bar{\lambda}(e^{-s}-1)A(\Delta T)(T_h - \Delta T)\right.
$$

$$
\left. + \frac{\bar{\lambda}(e^{-s}-1)(A\,\Delta T)^2(T_h - \Delta T)}{2}\right\}
$$

$$
= \exp\{[\bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)](e^{-s}-1)\}
$$

$$
\cdot \exp\left\{ [\bar{\lambda}\Delta T(A\,\Delta T) + \bar{\lambda}(T_h - \Delta T)(A\,\Delta T)^2]\,\frac{(e^{-s}-1)^2}{2}\right\}
$$

$$
\approx \exp\{[\bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)](e^{-s}-1)\}
\tag{22}
$$

which is a Poisson distribution with mean $\bar\lambda\Delta T + \bar\lambda T_h(A\,\Delta T)$. (The last approximation in Eq. (22) drops terms of order $O((\Delta T)^2)$). The final approximation has the same mean (Eq. (18)) as $\phi(s)$.

Example 2: Constant Light Source and Exponential Rate of Trapped Carrier Induced Events

Consider when $\lambda(t)$ is again constant as given by Eq. (6) and

$$h(t) = (aA)\,e^{-t/(T_h/a)} \tag{23}$$

(see Fig. 2) where A and T_h can be taken to be the same parameters as those in Eq. (15) for comparisons with the rectangular $h(t)$ of Example 1. Since T_h/a is the time constant in Eq. (23), the case where $a \gg 1$ is of most interest for comparison to the rectangular case. Also note that the areas under Eqs. (15) and (23) are both equal to AT_h.

It is easy to show that

$$\int_T^{T+\Delta T} h(t-\tau)\,dt = \begin{cases} AT_h\left(1 - e^{-\frac{\Delta T}{(T_h/a)}}\right)e^{-(T-\tau)/(T_h/a)}, & 0 \leqslant \tau \leqslant T \\[12pt] AT_h\left(1 - e^{-(T+\Delta T-\tau)/(T_h/a)}\right), & T \leqslant \tau \leqslant T+\Delta T \end{cases} \tag{24}$$

Substitution of Eq. (24) into Eqs. (3) through (5) yields

$$\phi(s) = \exp\left\{\bar\lambda \int_0^T [\exp\{-K_1\,e^{-\beta(T-\tau)}\} - 1]\,d\tau\right\}$$

$$\cdot\exp\left\{\bar\lambda \int_T^{T+\Delta T} [e^{-s}\exp\{-K_2 \right.$$

$$\left. \cdot(1 - e^{-\beta(T+\Delta T-\tau)})\} - 1]\,d\tau\right\} \tag{25}$$

where

$$K_1 = (1 - e^{-s})(AT_h)\left(1 - e^{-\frac{\Delta T}{T_h/a}}\right) \tag{26}$$

$$K_2 = (1 - e^{-s})(AT_h) \tag{27}$$

$$\beta = (T_h/a)^{-1} \tag{28}$$

The integrals in Eq. (25) can be expressed in terms of variants of the Exponential Integral function (Ref. 7, Chapter 5). It can be shown that for $s \geqslant 0$,

$$\phi(s) = \exp\left\{\frac{\bar\lambda}{\beta}[-E_1(K_1) + E_1(K_1\,e^{-\beta T}) - \beta T]\right\}$$

$$\cdot\exp\left\{\frac{\bar\lambda}{\beta}[e^{-(s+K_2)}\{Ei(K_2) - Ei(K_2\,e^{-\beta\Delta T})\} - \beta\Delta T]\right\} \tag{29}$$

and for $s < 0$,

$$\phi(s) = \exp\left\{\frac{\bar\lambda}{\beta}[Ei(-K_1) - Ei(-K_1\,e^{-\beta T}) - \beta T]\right\}$$

$$\cdot\exp\left\{\frac{\bar\lambda}{\beta}[e^{-(s+K_2)}\{-E_1(-K_2) + E_1(-K_2\,e^{-\beta\Delta T})\} \right.$$

$$\left. - \beta\Delta T]\right\} \tag{30}$$

where the Exponential Integral functions $E_1(\cdot)$ and $Ei(\ )$ are given in Ref. 7, Chapter 5. Finally substitution of Eq. (24) into Eqs. (10) and (11) yields

$$E[\Delta N_1 + \Delta N_2] =$$

$$\bar\lambda\Delta T + \bar\lambda T_h(A\Delta T) - e^{-\frac{T}{(T_h/a)}}\left(1 - e^{-\frac{\Delta T}{(T_h/a)}}\right)\left(\frac{\bar\lambda AT_h^2}{a}\right) \tag{31}$$

$$\text{var}(\Delta N_1 + \Delta N_2) =$$

$$E[\Delta N_1 + \Delta N_2]$$

$$+ \frac{(\bar\lambda T_h)(AT_h)^2}{2a}\left(1 - e^{-\frac{\Delta T}{T_h/a}}{}^2\right)\left(1 - e^{-\frac{2T}{(T_h/a)}}\right)$$

$$+ \bar\lambda\Delta T(AT_h)^2 + 2\bar\lambda\Delta T(AT_h)$$

$$- \frac{2\bar\lambda T_h(AT_h)(1+AT_h)}{a}\left(1 - e^{-\frac{\Delta T}{T_h/a}}\right)$$

$$+ \frac{(\bar\lambda T_h)(AT_h)^2}{2a}\left(1 - e^{-\frac{2\Delta T}{T_h/a}}\right) \tag{32}$$

Note that as $T \to \infty$, the average count given by Eq. (31) approaches the average count (Eq. (18)) in the rectangular $h(t)$ case. As $\Delta T \to 0$, the variance approaches the mean and so the dispersion again tends to that of a Poisson distribution. In fact, the distribution of $\Delta N_1 + \Delta N_2$ again tends to a

Poisson distribution as $\Delta T \to 0$. This can be established by using the approximations

$$E_1(x) = -\gamma - \ln x + x$$

$$Ei(x) = \gamma + \ln x + x$$

for small $x > 0$ (where γ is Euler's constant) in Eqs. (29) and (30), assuming that $aA\,\Delta T \ll 1$ and $\Delta T/(T_h/a) \ll 1$ and $T \to \infty$. Under these assumptions it can be shown that

$$\phi(s) \approx \exp\{[\bar{\lambda}\Delta T + \bar{\lambda}T_h(A\,\Delta T)]\,(e^{-s} - 1)\} \quad (33)$$

which is the same asymptotic Poisson distribution Eq. (22) as that in the rectangular $h(t)$ case when $\Delta T \to 0$. This is so because the primary events are Poisson distributed. So, given a fixed number of events in an interval, the occurrence times are uniformly distributed. This would then tend to average out the exponential $h(t)$ to produce an effective rectangular $h(t)$.

Example 3: Pulsed Light Source and Constant Rate of Trapped Carrier Induced Events

Consider a rectangular $h(t)$ given by Eq. (15) and a pulsed intensity rate $\lambda(t)$ given in Fig. 3. That is, $\lambda(t)$ is periodic of period T_p and pulse duration t_p and pulse height $\bar{\lambda}$. We assume that $T \geqslant \tau_p$ and $T + \Delta T \leqslant T_h$ and $T_p = \tau_p + T_h$. In this case, $\Delta N_1 + \Delta N_2$ will be the total count in any ΔT-second interval between pulses in steady state. So

$$\int_T^{T+\Delta T} h(t-\tau)\,dt = A\,\Delta T, \quad 0 \leqslant \tau \leqslant \tau_p$$

and hence Eqs. (3) through (5) yield

$$\phi(s) = \exp\{(\bar{\lambda}t_p)\,[\exp\{(e^{-s}-1)A\,\Delta T\} - 1]\} \quad (34)$$

This is the mgf of a two-parameter Neyman Type-A distribution. Here

$$E[\Delta N_1 + \Delta N_2] = (\bar{\lambda}\,\tau_p)\,(A\,\Delta T) \quad (35)$$

$$\text{var}(\Delta N_1 + \Delta N_2) = (1 + A\,\Delta T)\,E[\Delta N_1 + \Delta N_2] \quad (36)$$

Again, as $\Delta T \to 0$, $\phi(s)$ approaches a Poisson mgf with the same mean (Eq. (35)).

Example 4: Pulsed Light Source and Exponential Rate of Trapped Carrier Induces Events

Consider $\lambda(t)$ given as in Fig. 3 and an exponential $h(t)$ given by Eq. (23). We assume here $T \geqslant \tau_p$ and that $T_p \gg$

(T_h/a), the time constant of the exponential $h(t)$. We also assume that $T + \Delta T \leqslant T_p$. Then, as shown for Eqs. (24) and (25), it can be shown that

$$\phi(s) = \exp\left\{\bar{\lambda}\int_{T-\tau_p}^{T} [\exp\{-K_1\,e^{-\beta x}\} - 1]\,dx\right\} \quad (37)$$

where K_1 and β are given by Eqs. (26) and (28), respectively. The integral in Eq. (37) can be expressed in terms of Exponential Integral functions as in Eqs. (29) and (30).

In fact,

$$\phi(s) = \begin{cases} \exp\left\{\dfrac{\bar{\lambda}}{\beta}\,[-E_1(K_1\,e^{-\beta(T-\tau_p)}) + E_1(K_1\,e^{-\beta T}) - \beta\tau_p]\right\}, \\ \quad \text{if } s \geqslant 0 \\[1em] \exp\left\{\dfrac{\bar{\lambda}}{\beta}\,[Ei(-K_1\,e^{-\beta(T-\tau_p)}) - Ei(-K_1\,e^{-\beta T}) - \beta\tau_p]\right\}, \\ \quad \text{if } s < 0 \end{cases}$$

$$(38)$$

It can also be shown that

$$E[\Delta N_1 + \Delta N_2] =$$

$$\frac{(\bar{\lambda}T_h)\,(AT_h)}{a}\left(1 - e^{-\frac{\Delta T}{(T_h/a)}}\right)\left(e^{\frac{\tau_p}{T_h/a}} - 1\right)e^{-\frac{T}{(T_h/a)}}$$

$$(39)$$

and

$$\text{var}(\Delta N_1 + \Delta N_2) =$$

$$E[\Delta N_1 + \Delta N_2]$$

$$+\frac{\bar{\lambda}T_h(AT_h)^2}{2a}\left(1 - e^{-\frac{\Delta T}{T_h/a}}\right)^2\left(e^{\frac{2\tau_p}{(T_h/a)}} - 1\right)e^{-\frac{2T}{(T_h/a)}}$$

$$(40)$$

Note that as $\Delta T \to 0$, the variance approaches the mean. In fact, it can be shown that as $\Delta T \to 0$, $\phi(s)$ approaches a Poisson mgf with mean given by

$$(\bar{\lambda}T_h)\,(A\,\Delta T)\left(e^{\frac{\tau_p}{T_h/a}} - 1\right)e^{-\frac{T}{T_h/a}} \quad (41)$$

IV. Conclusions

In this article, moment generating functions of the output count statistics of a TAD-mode APD were derived under the assumptions that the primary avalanche events are Poisson distributed and the secondary trapped-carrier-induced avalanche events generated by each primary event are conditionally Poisson. The mean and variance of the distributions were also derived. Examples of constant light source and pulsed light source excitation and constant and exponential decaying rate of secondaries generated per primary were considered. It is shown that the dispersion of the distribution is always larger than that of the Poisson. In each of the examples, the distribution approached a Poisson distribution when the count interval tended to zero.

We should note that trapped carriers generated by a particular primary event were assumed to be unaffected by future avalanche events. It was also assumed that trapped-carrier-induced avalanche events do not themselves produce more trapped carriers. These simplifying assumptions were made in the interest of analytical tractability. Hence the usefulness of these results need to be established experimentally.

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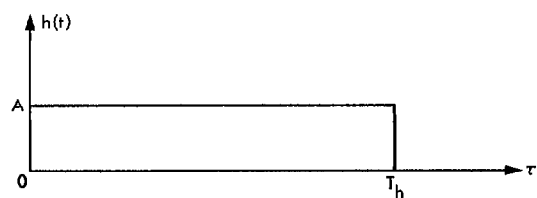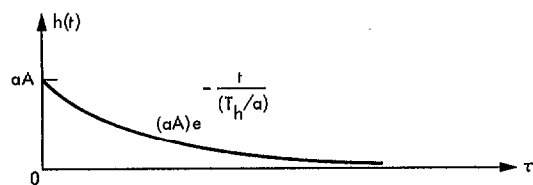
Fig. 1. Rectangular *h*(*t*)
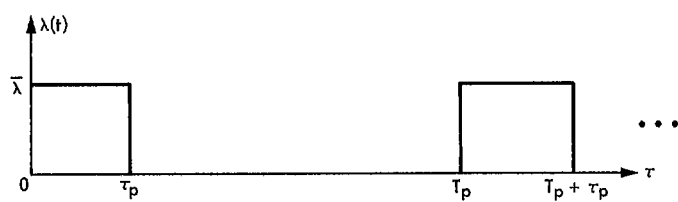

Fig. 2. Exponential *h*(*t*)

Fig. 3. Pulsed light λ(*t*)

Appendix A

Derivation of $\phi(s)$

First we note that

$$
\phi(s) = \sum_{k=0}^{\infty} e^{-sk} P[\Delta N_1 + \Delta N_2 = k]
$$

$$
= \sum_{k=0}^{\infty} e^{-sk} \sum_{i=0}^{\infty} \sum_{\ell=0}^{k} P[\Delta N_1 + \Delta N_2 = k, \Delta N_1 = \ell, N_1(T) = i]
$$

$$
\overset{(1)}{=} \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} e^{-sk} \left\{ \sum_{i=0}^{\infty} P[\Delta N_2 = k - \ell \mid \Delta N_1 = \ell, N_1(T) = i] \cdot P[\Delta N_1 = \ell] \cdot P[N_1(T) = i] \right\}
$$

$$
\overset{(2)}{=} \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} e^{-s(j+\ell)} \left\{ \sum_{i=0}^{\infty} P[\Delta N_2 = j \mid \Delta N_1 = \ell, N_1(T) = i] \cdot P[\Delta N_1 = \ell] \cdot P[N_1(T) = i] \right\}
$$

$$
= \sum_{i=0}^{\infty} P[N_1(T) = i] \left[\sum_{\ell=0}^{\infty} e^{-s\ell} P[\Delta N_1 = \ell] \cdot \left\{ \sum_{j=0}^{\infty} e^{-sj} P[\Delta N_2 = j \mid \Delta N_1 = \ell, N_1(T) = i] \right\} \right]
$$

$$\qquad\qquad (A-1)$$

where Subequation (1) is because $N_1(t)$ has independent increments and Subequation (2) is obtained by setting $j = k - 1$ and noting that

$$
\sum_{k=0}^{\infty} \sum_{\ell=0}^{k} = \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty}
$$

Next consider $i \geqslant 1$ and $\ell \geqslant 1$ and suppose that $N_1(T) = i$ and $\Delta N_1 = \ell$. Then let $t_1, \cdots, t_i$ be the primary event occurrence times in $[0, T)$ and $t_{i+1}, \cdots, t_{i+\ell}$ the occurrence times in $[T, T + \Delta T)$. Then

$$
\sum_{j=0}^{\infty} e^{-sj} P[\Delta N_2 = j \mid \Delta N_1 = \ell, N_1(T) = i] = E_{t_1, \cdots, t_{i+\ell}} \left[\sum_{j=0}^{\infty} e^{-sj} \frac{\left(\sum_{m=1}^{i+\ell} \int_T^{T+\Delta T} h(t - t_m) \, dt \right)^j}{j!} \right.
$$

$$
\left. \cdot \exp\left(- \sum_{m=1}^{i+\ell} \int_T^{T+\Delta T} h(t - t_m) \, dt \right) \cdot \mid \Delta N_1 = \ell, N_1(T) = i \right]
$$

$$\qquad\qquad (A-2)$$

$$
= E_{t_1, \cdots, t_{i+\ell}} \left[\prod_{m=1}^{i+\ell} \exp\left\{ (e^{-s} - 1) \int_T^{T+\Delta T} h(t - t_m) \, dt \right\} \mid \Delta N_1 = \ell, N_1(T) = i \right]
$$

$$\overset{(1)}{=} E_{t_1,\cdots,t_i}\left[\prod_{m=1}^{i}\exp\left\{(e^{-s}-1)\int_{T}^{T+\Delta T}h(t-t_m)\,dt\right\}\,|\,N_1(T)=i\right]$$

$$\cdot\,E_{t_{i+1},\cdots,t_{i+\ell}}\left[\prod_{m=i+1}^{i+\ell}\exp\left\{(e^{-s}-1)\int_{T}^{T+\Delta T}h(t-t_m)\,dt\right\}\,|\,\Delta N=\ell\right]$$

$$\overset{(2)}{=} K^i\cdot H^\ell \tag{A-2}$$

where

$$K=\frac{\displaystyle\int_0^T\lambda(\tau)\exp\left[(e^{-s}-1)\int_T^{T+\Delta T}h(t-\tau)\,dt\right]d\tau}{\displaystyle\int_0^T\lambda(\tau)\,d\tau} \tag{A-3}$$

and

$$H=\frac{\displaystyle\int_T^{T+\Delta T}\lambda(\tau)\exp\left[(e^{-s}-1)\int_T^{T+\Delta T}h(t-\tau)\,dt\right]d\tau}{\displaystyle\int_T^{T+\Delta T}\lambda(\tau)\,d\tau} \tag{A-4}$$

The independent increments property of $N_1(t)$ is used to obtain Subequation (1) in Eq. (A-2) and follows from known properties of the conditional distribution of occurrence times, given the number of events in a fixed time interval for a Poisson process (Ref. 4).

Next, we note from the derivation that the final result in Eq. (A-2) is valid when either $i=0$ or $\ell=0$ or both $i=\ell=0$. Finally, substitution of Eqs. (A-2), (A-3), and (A-4) into Eq. (A-1) yields:

$$\phi(s)=\left\{\sum_{i=0}^{\infty}K^2\,P[N_1(T)=i]\right\}\left\{\sum_{\ell=0}^{\infty}(He^{-s})^\ell\,P[\Delta N_1=\ell]\right\}$$

$$=\left\{\sum_{i=0}^{\infty}K^i\frac{\left(\int_0^T\lambda(\tau)\,d\tau\right)^i}{i!}\exp\left(-\int_0^T\lambda(\tau)\,d\tau\right)\right\}$$

$$\cdot\left\{\sum_{\ell=0}^{\infty}(He^{-s})^\ell\frac{\left(\int_T^{T+\Delta T}\lambda(\tau)\,d\tau\right)^\ell}{\ell!}\exp\left(-\int_T^{T+\Delta T}\lambda(\tau)\,d\tau\right)\right\} \tag{A-5}$$

$$= \exp \left\{ \int_0^T \lambda(\tau) \left[\exp \left\{ (e^{-s} - 1) \int_T^{T+\Delta T} h(t - \tau) \, dt \right\} - 1 \right] dt \right\}$$

$$\cdot \exp \left\{ \int_T^{T+\Delta T} \lambda(\tau) \left[e^{-s} \exp \left\{ (e^{-s} - 1) \int_T^{T+\Delta T} h(t - \tau) \, dt \right\} - 1 \right] d\tau \right\} \Bigg\} \quad \text{(A-5)}$$